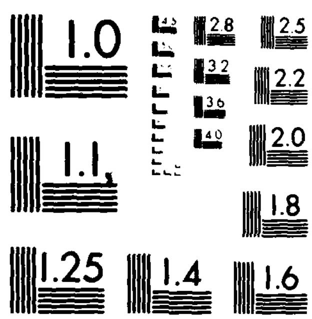


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# FREE BOUNDARY PROBLEM FOR UNSTEADY SLAG FLOW IN THE HEARTH

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# FREE BOUNDARY PROBLEM FOR UNSTEADY SLAG FLOW IN THE HEARTH

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## ABSTRACT

Drainage of the hearth is an important component of a successful operation of a blast furnace. In order to investigate the influence of tapping conditions due to the shape of the slag surface, the one-phase flow of slag during tapping was modeled as a *free boundary problem*. This problem was reformulated by using the method of integrated penalty and, then was simulated by using a finite difference method developed by the author.

The objective of this report is to give a mathematical justification of the penalty method formulation, in which the perturbation with respect to the domain and the asymptotic properties of solutions of the boundary value problem for an elliptic equation with penalty terms are used.


AMS (MOS) Subject Classifications: 34E05, 34E99, 35J05, 35J67, 35R35

Key Words: Free boundary problem, Method of integrated penalty, Asymptotic expansions, Perturbation with respect to domain

Work Unit Number 1 (Applied Analysis)

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## SIGNIFICANCE AND EXPLANATION

We can observe many phenomena involving Free Boundaries in various fields of engineering and applied science, for example, jet problems, transient multi-fluid flows, the equilibria of plasmas, Stefan problems, free boundary problems in optimal shape designs and others. Hence there is interest and need to develop efficient and accurate numerical methods for the solution of these problems. Some free boundary problems mentioned above have been successfully solved by using the penalty method developed by the author and his colleagues. An important feature of our approach is that the outward normal derivative of the solution at the free boundary is approximated efficiently.

The objective of this report is to give the required mathematical justification for the model problem reformulated by the method of integrated penalty. Specifically, we prove the convergence of the penalized free boundary to the original one as the penalizing parameter $\varepsilon$ tends to zero. Here the perturbation theory with respect to the domain and the asymptotic properties of the solutions of a boundary value problem for an elliptic equation with penalty terms are used. As an application, the unsteady slag flow in the hearth is considered.



The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

# FREE BOUNDARY PROBLEM FOR UNSTEADY SLAG FLOW IN THE HEARTH

Hideo Kawarada*

## 1. INTRODUCTION

The hearth drainage is one of the most important factors for successful blast furnace operation. The slag is considered to be more difficult to drain than the metal because of its higher viscosity. When the slag surface reaches the level of tap hole, the furnace gas starts to blow out. Then tapping should be stopped. The amount of undrained molten material at the end of tapping is estimated by the shape of the slag surface. In order to determine the influence of tapping conditions due to the shape of the slag surface, the three-dimensional problem of the slag flow during tapping was solved by using the finite element method by Ichihara and Fukutake [2]. They concluded that their computation scheme is not efficient in practical use. This computational instability was resolved by Kawarada and Natori [11], using the penalty method developed by themselves [5-7, 10].

The objective of this report is to give mathematical justification of penalty formulation, i.e., to prove the convergence of the penalized free boundary to the one of an original problem when we let the penalizing parameter $\varepsilon$ tend to zero. In section 2, we review the formulation for two-dimensional problems of the slag flow. In section 3, we give the penalized formulation by using the method of integrated penalty. Section 4 is devoted to the assumptions and the main theorem. In section 5, we prepare some propositions needed to prove the main theorem, in which the perturbation of the penalized solution with respect to the domain is discussed. Finally, we give the proof of the main theorem in section 6.

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## 2. FORMULATION

We consider two-dimensional problem of the slag flow in the hearth, which is bounded by impermeable boundaries $y = 0$, $x = 0$ and $x = a$ (c.f. Figure 1). One of the vertical boundaries, $x = 0$, has a tapping hole near the bottom, which we denote by $\Gamma_0$.

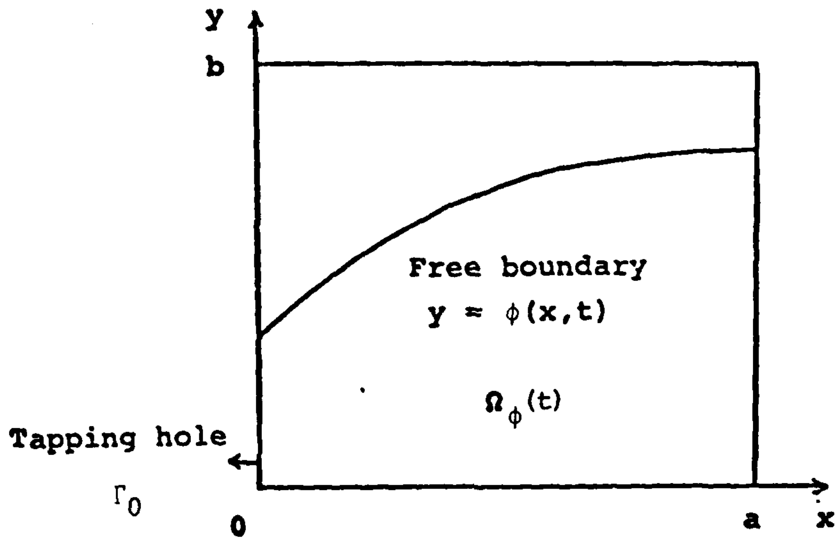


Figure 1

We assume that the hearth is packed with coke, through the bed of which the slag flows. Then Darcy's law can be applied for the flow of slag:

$$V = -d\nabla\Phi \tag{2.1}$$

where $V$ denotes velocity of slag. The potential $\Phi$ is defined as follows:

$$\Phi = \frac{p - p_0}{\rho g} + y \tag{2.2}$$

where $\rho$: density of slag,

$g$: gravitational acceleration,

$p$: pressure of slag,

$p_0$: pressure at reference point,

$y$: vertical height from a horizontal plane,

$d$: permeability of slag.

If we substitute $V$ into the equation of continuity:

$$\text{(2.3)} \qquad \operatorname{div} V = 0 ,$$

then we have

$$\text{(2.4)} \qquad \Delta\Phi = 0 \text{ in the slag region } \Omega_\phi(t) \quad (t \in [0,T)) ,$$

under the condition $d$ = constant and $T$ is tapping period. $\Omega_\phi(t)$ is defined as follows:

$$\text{(2.5)} \qquad \Omega_\phi(t) = \{(x,y) \mid 0 < x < a,\ 0 < y < \phi(x,t)\}$$

where $y = \phi(x,t)$ represents the height of the slag surface, which is a free boundary. The boundary conditions for the potential $\Phi$ are given:

(2.6) $\Phi_y = 0$ on $y = 0$ ,

(2.7) $\Phi_x = 0$ on $x = 0$ *and* $x = a$, except on the tapping hole $\Gamma_0$ ,

(2.8) $\Phi_x = V_{out}$ on $\Gamma_0$ ,

(2.9) $\Phi = y$ on $y = \phi(x,t)$ .

The drainage rate $V_{out}$ of a working blast furnace increases as the tap hole is eroded during tapping:

$$V_{out} = \bar{V}_{out}\left(k + \ell \cdot \frac{t}{T}\right) \qquad (0 \leq t \leq T)$$

where $\bar{V}_{out}$: average drainage rate,

$k$: tap hole opening rate,

$\ell$: tap hole erosion rate.

$$\text{(2.11)} \qquad \frac{\partial\phi}{\partial t} = d(\Phi_x\phi_x - \Phi_y)\Big|_{y=\phi(x,t)} + V_{in} ,$$

$$= -d\cdot\sqrt{1 + \phi_x^2} \cdot \frac{\partial\Phi}{\partial n}\Big|_{y=\phi(x,t)} + V_{in} .$$

Where $n$ is outward normal to $\Omega_{\phi}(t)$ and $V_{in}$ is the inflow velocity of slag at the surface. The initial shape of the free surface is given:

$$\phi(x,0) = \phi_0(x) \qquad (0 < x < a) \ . \tag{2.12}$$

In this report, we deal with the case $V_{out}$ is independent of $t$, but dependent of $y$. Hereafter, we denote the free boundary problem (2.4)-(2.12) by (P). By using the relation (2.2), (P) is reformulated into the equations for $\{p$ and $y = \phi\}$:

$$\Delta p = 0 \qquad \text{in } \Omega_{\phi}(t) \qquad (0 \leqq t \leqq T) \ , \tag{2.13}$$

$$\frac{\partial p}{\partial y} = -\rho g \qquad \text{on } y = 0 \ , \tag{2.14}$$

$$\frac{\partial p}{\partial x} = 0 \qquad \text{on } x = 0 \text{ and } x = a \text{ except on } \Gamma_0 \ , \tag{2.15}$$

$$\frac{\partial p}{\partial x} = \rho g \cdot V_{out} \qquad \text{on } \Gamma_0 \ , \tag{2.16}$$

$$p = p_0 \qquad \text{on } y = \phi(x,t) \ , \tag{2.17}$$

$$\frac{\partial \phi}{\partial t} = -d \cdot \left(1 + \frac{1}{\rho g} \sqrt{1 + \phi_x^2} \cdot \left.\frac{\partial p}{\partial n}\right|_{y=\phi}\right) + V_{in} \ , \tag{2.18}$$

$$\phi(x,0) = \phi_0(x) \qquad (0 < x < a) \ . \tag{2.19}$$

For simplicity, we take $\rho g = 1$ and $p_0 = 0$ and denote the above problem by (P').

## 3. AN APPROXIMATION OF (P') BY MEANS OF THE METHOD OF INTEGRATED PENALTY

When we try to solve (P'), the numerical procedure must contain a routine for solving the potential problem (2.13)-(2.17) for a given free boundary $y = \phi(x,t)$. After this is done, the outward normal derivative of the potential on the free boundary can be calculated. And then, by solving (2.18) and (2.19), the subsequent shape of the free boundary is obtained and so on. If we apply the method of integrated penalty to solve the potential problem (2.13)-(2.14), then the outward normal derivative of the potential function on the free boundary are easily approximated [3, 10]. This is the reason why we apply the method of integrated penalty to free boundary problems.

Let $B = \{(x,y) | 0 < x < a,\ 0 < y < b\}$ and $Y(t)$ $(t \in \mathbb{R})$ be the heaviside function. Then we penalize (P') as follows: Find $\{p^\varepsilon$ and $y = \phi^\varepsilon\}$ for $\forall\ \varepsilon > 0$,

$$\Delta p^\varepsilon - \frac{1}{\varepsilon} \cdot Y(y - \phi^\varepsilon(x,t))p^\varepsilon = 0 \quad \text{in} \quad B , \tag{3.1}$$

$$\frac{\partial p^\varepsilon}{\partial x} = 0 \qquad \text{on} \quad x = 0 \quad \text{and} \quad x = a, \quad \text{except on} \quad \Gamma_0 , \tag{3.2}$$

$$\frac{\partial p^\varepsilon}{\partial x} = V_{out} \quad \text{on} \quad \Gamma_0 , \tag{3.3}$$

$$\frac{\partial p^\varepsilon}{\partial y} = -1 \qquad \text{on} \quad y = 0 , \tag{3.4}$$

$$p^\varepsilon = 0 \qquad \text{on} \quad y = b . \tag{3.5}$$

Obviously, $Y(y - \phi^\varepsilon(x,t))$ $(0 < x < a,\ 0 < t < T)$ is the characteristic function of $B - \Omega_{\phi^\varepsilon}(t)$, where

$$\Omega_{\phi^\varepsilon}(t) = \{(x,y) | 0 < x < a,\ 0 < y < \phi^\varepsilon(x,t)\} .$$

It is well known that if we let $\varepsilon$ be small enough in (3.1), then $p^\varepsilon$ approximates $p$ in $\Omega_{\phi^\varepsilon}(t)$ and $p^\varepsilon$ is nearly equal to zero in $B - \Omega_{\phi^\varepsilon}(t)$

[9]. Therefore the boundary condition (2.17) is approximately satisfied. If we use the method of the integrated penalty, the equation (2.18) is approximated by

$$\frac{\partial\phi^\varepsilon}{\partial t} = -d\left\{1 - \frac{1}{\varepsilon}\int_{\phi^\varepsilon}^{b} p^\varepsilon(x,\eta)d\eta\right\} + V_{in} . \tag{3.6}$$

This approximation is based on the following fact; put

$$r^\varepsilon = \frac{1}{\varepsilon}\cdot Y(y - \phi^\varepsilon)p^\varepsilon , \tag{3.7}$$

$$s^\varepsilon = \int_{y}^{b} r^\varepsilon(x,\eta,t)d\eta . \tag{3.8}$$

By an application of Theorems 1.1 and 1.2 in [3], we have

$$r^\varepsilon \to \sqrt{1+\phi_x^2}\cdot\left.\frac{\partial p}{\partial n}\right|_{y=\phi(x,t)}\cdot\frac{\partial Y}{\partial y} \quad \text{in} \quad \mathcal{D}'(B) , \tag{3.9}$$

$$s^\varepsilon \to -\sqrt{1+\phi_x^2}\cdot\left.\frac{\partial p}{\partial n}\right|_{y=\phi(x,t)}\cdot(1 - Y) \quad \text{in} \quad \mathcal{D}'(B) \quad \text{as} \quad \varepsilon \to 0 . \tag{3.10}$$

By using (3.10), we have

$$-\sqrt{1+\phi_x^2}\cdot\left.\frac{\partial p}{\partial n}\right|_{y=\phi(x,t)} \simeq s^\varepsilon(x,\phi^\varepsilon) = \frac{1}{\varepsilon}\int_{\phi^\varepsilon}^{b} p^\varepsilon(x,\eta)d\eta \tag{3.11}$$

in a suitable topology. Substituting (3.11) into (2.18), we have (3.6). Now, $s^\varepsilon(x,\phi^\varepsilon)$ is called the integrated penalty. Hereafter we denote by $(P^\varepsilon)$ the penalized problem defined above. Finally we should note that there holds the same discussion about (P) as mentioned above.

## 4. THE ASSUMPTIONS AND THE MAIN RESULT

4.1. In order to obtain our main result, we need the following assumptions:

Assumption (A)

There exists a unique solution $\{p \text{ and } y = \phi(x,t)\}$ of $(P')$ such that

$$p \in C^{3,\alpha}(\Omega_\phi(t)) \quad \text{for } t \in (0,T] \text{ and } \alpha \in (0,1)\ , \tag{4.1}$$

$$\phi \in C^{3,\alpha}(0,a) \quad \text{for } t \in (0,T] \text{ and } \alpha \in (0,1)\ , \tag{4.2}$$

$$\phi \in C^{1}(0,T) \quad \text{for } x \in (0,a)\ . \tag{4.3}$$

Naturally, $V_{out}$ included in the boundary condition should have regularity property in order to obtain (4.1).

Assumption (B)

There exists a unique solution $\{p^\varepsilon \text{ and } y = \phi^\varepsilon(x,t)\}$ of $(P^\varepsilon)$ such that

$$p^\varepsilon \in C^{3,\alpha}(\Omega_{\phi^\varepsilon}(t)) \quad \text{for } t \in (0,T] \text{ and } \alpha \in (0,1)\ , \tag{4.4}$$

$$\phi^\varepsilon \in C^{3,\alpha}(0,a) \quad \text{for } \alpha \in (0,1)\ , \tag{4.5}$$

$$\phi^\varepsilon \in C^{1}(0,T) \quad \text{for } x \in (0,a)\ , \tag{4.6}$$

moreover

$$|\phi^\varepsilon|_{C^{3,\alpha}(0,a)} \leq K \tag{4.7}$$

where $K$ is independent of $\varepsilon$.

4.2. Then we have the main result.

Theorem. Under the assumptions (A) and (B), let $\varepsilon \to 0$, then

$$\phi^\varepsilon \to \phi \text{ uniformly in } (0,a) \times (0,T]. \tag{4.8}$$

## 5. PREPARATIONS

In order to prove our main theorem, we have to prepare three propositions.

5.1. We consider the following penalized problem defined in $B$: For $\forall_\varepsilon > 0$,

$$-\Delta p^\varepsilon + \frac{1}{\varepsilon} \cdot Y(y - \phi(x)) \cdot p^\varepsilon = 0 \quad \text{in} \quad B , \tag{5.1}$$

$$\frac{\partial p^\varepsilon}{\partial x} = 0 \qquad \text{on} \quad x = 0 \quad \text{and} \quad x = a, \quad \text{except on} \quad \Gamma_0 , \tag{5.2}$$

$$\frac{\partial p^\varepsilon}{\partial x} = V_{out} \quad \text{on} \quad \Gamma_0 , \tag{5.3}$$

$$\frac{\partial p^\varepsilon}{\partial y} = -1 \qquad \text{on} \quad y = 0 , \tag{5.4}$$

$$p^\varepsilon = 0 \qquad \text{on} \quad y = b . \tag{5.5}$$

Where $y = \phi(x)$ is smooth in $(0,a)$ and satisfies

$$0 < \phi(x) < b \quad (0 < x < a) . \tag{5.6}$$

Define

$$\Omega_\phi = \{(x,y) | 0 < x < a, \ 0 < y < \phi(x)\}$$

and

$$\Gamma = \{(x,y) | 0 < x < a, \ y = \phi(x)\} .$$

Let $p^0$ be the solution of the problem:

$$\Delta p^0 = 0 \qquad \text{in} \quad \Omega_\phi , \tag{5.7}$$

$$\frac{\partial p^0}{\partial x} = 0 \qquad \text{on} \quad x = 0 \quad \text{and} \quad x = a, \quad \text{except on} \quad \Gamma_0 , \tag{5.8}$$

$$\frac{\partial p^0}{\partial x} = V_{out} \quad \text{on} \quad \Gamma_0 , \tag{5.9}$$

$$p^0 = 0 \qquad \text{on} \quad \Gamma \quad \text{and} \quad \frac{\partial p^0}{\partial y} = -1 \quad \text{on} \quad y = 0 . \tag{5.10}$$

It is well known [9] that

(5.11) $p^{\varepsilon} \to p^{0} \cdot (1 - Y)$ strongly in $H^{1}(B)$ as $\varepsilon \to 0$ .

Then we have

Proposition 1. Let $\varepsilon$ be small enough and $m > 0$. Then

$$\|p^{\varepsilon}\|_{m,\Gamma} \leq O(\sqrt{\varepsilon})\ \left\|\frac{\partial p^{0}}{\partial n}\right\|_{m,\Gamma}, \quad (*) \tag{5.12}$$

$$\left\|p^{\varepsilon} + \sqrt{\varepsilon} \cdot \frac{\partial p^{0}}{\partial n}\right\|_{m,\Gamma} \leq O(\varepsilon)\ \left\|\frac{\partial p^{0}}{\partial n}\right\|_{m,\Gamma}, \tag{5.13}$$

$$\left\|\int_{\Gamma^{\perp}} p^{\varepsilon} d\Gamma^{\perp} - \sqrt{\varepsilon} \cdot p^{\varepsilon}\right\|_{m,\Gamma} \leq O(\varepsilon)\|p^{\varepsilon}\|_{m+1,\Gamma} . \tag{5.14}$$

Proof. See Kawarada and Hanada ([4], Chapter 2, p. 4).

5.2. We fix some $y = \phi^{0}(x)$, which satisfies (5.6) and is smooth enough and we denote the problem (5.7)-(5.10) by $P_0$. Then any domain $\Omega_{\phi}$ which is diffeomorphic to $\Omega_{\phi^0}$ can be defined enough near $\Omega_{\phi^0}$ such that

$$\begin{aligned} \Omega_{\phi} &= \{(x,y)\,|\,0 < x < a,\ 0 < y < \phi(x)\} \\ &\text{where } \phi(x) = \phi^{0}(x) + \delta\phi(x) \qquad (0 < x < a) \\ &\text{with } \delta\phi \in C^{\mu}(0,a) \qquad (\mu > 1) . \end{aligned} \tag{5.15}$$

We consider a family of the penalized problem (5.1)-(5.5), in which $Y(y - \phi(x))$ is replaced by $Y(y - \phi'(x))$ $(\phi' = \phi + \delta\phi)$. Denote the solution of this problem by $p^{\varepsilon'}$. Then

Proposition 2.

$$\frac{\partial p^{\varepsilon}}{\partial \phi} = \lim_{\|\delta\phi\|_{C^{\mu}(0,a)} \to 0} \frac{p^{\varepsilon'} - p^{\varepsilon}}{\delta\phi} \text{ exists ,} \tag{5.16}$$

$$\frac{\partial p^{\varepsilon}}{\partial \phi} \in C^{\alpha}(B) \qquad (\alpha \in (0,1)) . \tag{5.17}$$

(*) $\|V\|_{m,G}$ indicates the norm of $V$ in $H^{m}(G)$.

Proof. See Dervieux ([1], Chapter 3, p. 47). Also $\frac{\partial p^\varepsilon}{\partial \phi}$ satisfies the same properties as $p^\varepsilon$ does in Proposition 1.

Proposition 3.

$$\left\| \frac{\partial p^\varepsilon}{\partial \phi} - \frac{1}{\sqrt{\varepsilon}} p^\varepsilon \langle n, e_y \rangle \right\|_{m,\Gamma} \leq O(\sqrt{\varepsilon}) \| \langle n, e_y \rangle \|_{m+1,\infty,\Gamma} \times \left\| \frac{\partial p^0}{\partial n} \right\|_{m+1,\Gamma} \tag{5.18}$$

$$\left\| \int_{\Gamma^\perp} \frac{\partial p^\varepsilon}{\partial \phi} d\Gamma^\perp - \sqrt{\varepsilon} \frac{\partial p^\varepsilon}{\partial \phi} \right\|_{m,\Gamma} \leq O(\varepsilon) \| \langle n, e_y \rangle \|_{m+1,\infty,\Gamma} \times \left\| \frac{\partial p^0}{\partial n} \right\|_{m+1,\Gamma} \tag{5.19}$$

where $n$ is unit outward normal to $\Omega_\phi, e_y = \nabla y = (0,1)$, $\langle \cdot, \cdot \rangle$ implies the inner product in $\mathbb{R}^2$ and $\|v\|_{m,\infty,\Gamma}$ indicates the norm of $v$ in $W^{m,\infty}(\Gamma)$.

Proof. $p^\varepsilon$ satisfies

$$-\Delta p^\varepsilon + \frac{1}{\varepsilon} \cdot Y(y - \phi(x)) p^\varepsilon = 0 \quad \text{in} \quad B . \tag{5.20}$$

Differentiating both sides of (5.20) with respect to $\phi$, we have

$$-\Delta \frac{\partial p^\varepsilon}{\partial \phi} + \frac{1}{\varepsilon} Y(y - \phi) \frac{\partial p^\varepsilon}{\partial \phi} - \frac{1}{\varepsilon} \cdot \frac{\partial Y}{\partial y} \cdot p^\varepsilon = 0 \quad \text{in} \quad B . \tag{5.21}$$

Put $Q^\varepsilon = \frac{\partial p^\varepsilon}{\partial \phi}$, multiply $\psi \in H_0^1(B)$ both sides of (5.21) and integrate in B. Then we have

$$\iint_B \nabla Q^\varepsilon \cdot \nabla \psi \, dxdy + \frac{1}{\varepsilon} \iint_B Y Q^\varepsilon \psi \, dxdy - \frac{1}{\varepsilon} \iint_B \frac{\partial Y}{\partial y} p^\varepsilon \psi \, dxdy = 0 . \tag{5.22}$$

By using the relation;

$$\iint_B \frac{\partial Y}{\partial y} p^\varepsilon \psi \, dxdy = \int_\Gamma p^\varepsilon \langle n, e_y \rangle \psi \, d\Gamma , \tag{5.23}$$

we have

$$\iint_B (-\Delta + \frac{1}{\varepsilon}\,\gamma)Q^{\varepsilon}\cdot\psi\,dxdy + \int_P \left(\frac{\partial Q_0^{\varepsilon}}{\partial n} - \frac{\partial Q_1^{\varepsilon}}{\partial n} - \frac{1}{\varepsilon}\cdot p^{\varepsilon}\cdot\langle n, e_y\rangle\right)\psi\,d\Gamma = 0 \tag{5.24}$$

(by Green's formula) where $Q_0^{\varepsilon} = Q^{\varepsilon}|_{\Omega_\phi}$ and $Q_1^{\varepsilon} = Q^{\varepsilon}|_{B-\Omega_\phi}$. (5.24) implies

$$\Delta Q_0^{\varepsilon} = 0 \quad \text{in } \Omega_\phi , \tag{5.25}$$

$$-\Delta Q_1^{\varepsilon} + \frac{1}{\varepsilon} Q_1^{\varepsilon} = 0 \quad \text{in } B - \Omega_\phi , \tag{5.26}$$

$$Q_0^{\varepsilon} = Q_1^{\varepsilon} \quad \text{on } \Gamma \tag{5.27}$$

$$\frac{\partial Q_0^{\varepsilon}}{\partial n} - \frac{\partial Q_1^{\varepsilon}}{\partial n} = \frac{1}{\varepsilon}\cdot p^{\varepsilon}\cdot\langle n, e_y\rangle \quad \text{on } \Gamma \tag{5.28}$$

$$\frac{\partial Q_i^{\varepsilon}}{\partial x} = 0 \quad \text{on } x = 0 \text{ and } x = a, \text{ except on } \Gamma_0 \ (i = 0,1) , \tag{5.29}$$

$$\frac{\partial Q_0^{\varepsilon}}{\partial x} = \frac{\partial v_{out}}{\partial y} \quad \text{on } \Gamma_0 , \tag{5.30}$$

$$\frac{\partial Q_0^{\varepsilon}}{\partial y} = 0 \quad \text{on } y = 0 , \tag{5.31}$$

$$Q_1^{\varepsilon} = 0 \quad \text{on } y = b . \tag{5.32}$$

(5.27) comes from (5.17). If we use the method developed in Kawarada and Hanada ([4], Chapter 5, p. 15), we are able to obtain (5.18) and (5.19).

## 6. THE PROOF OF THE MAIN THEOREM

6.1. We reformulate (2.18) into the following penalty formulation by means of the method of integrated penalty:

$$\frac{\partial\phi}{\partial t} = -d\left(1 - \frac{1}{\varepsilon}\int_{\phi}^{b} q^{\varepsilon}(x,\eta)d\eta\right) + V_{in} + O(\sqrt{\varepsilon}) \tag{6.1}$$

$q^{\varepsilon}$ is the solution of the problem:

$$-\Delta q^{\varepsilon} + \frac{1}{\varepsilon}\cdot Y(y - \phi(x,t))\cdot q^{\varepsilon} = 0 \quad \text{in} \quad B \tag{6.2}$$

and $q^{\varepsilon}$ satisfies the same boundary conditions as $p^{\varepsilon}$ does on $\partial B$. In order to justify (6.1), we may show the following inequality:

$$\left\|\frac{1}{\varepsilon}\int_{\phi}^{b} q^{\varepsilon}d\eta + \sqrt{1+\phi_x^2}\cdot\frac{\partial p}{\partial n}\right\|_{C(\Gamma)}$$

$$\leq \left\|\sqrt{1+\phi_x^2}\right\|_{C(0,a)}\cdot\left\|\frac{1}{\varepsilon}\int_{\Gamma^{\perp}} q^{\varepsilon}d\Gamma^{\perp} + \frac{\partial p}{\partial n}\right\|_{C(\Gamma)}$$

$$\leq \left\|\sqrt{1+\phi_x^2}\right\|_{C(0,a)}\cdot\left\|\frac{1}{\varepsilon}\int_{\Gamma^{\perp}} q^{\varepsilon}d\Gamma^{\perp} + \frac{\partial p}{\partial n}\right\|_{H^1(\Gamma)}$$

$$\leq \left\|\sqrt{1+\phi_x^2}\right\|_{C(0,a)}\cdot O(\sqrt{\varepsilon})\left\|\frac{\partial p}{\partial n}\right\|_{H^2(\Gamma)} \quad \text{(by (5.13) and (5.14))}$$

$$\leq O(\sqrt{\varepsilon})\left\|\sqrt{1+\phi_x^2}\right\|_{C(0,a)}\cdot\left\|\frac{\partial p}{\partial n}\right\|_{C^2(\Gamma)}$$

$$\leq O(\sqrt{\varepsilon})\left\|\sqrt{1+\phi_x^2}\right\|_{C(0,a)}\cdot\|p\|_{C^3(\Omega_{\phi}(t))} \quad (t \in (0,T]) .$$

If we use the regularity property of the solution of an elliptic boundary value problem (see Theorem 3.3.1 in [8]), we have

$$\|p\|_{C^{3,\alpha}(\Omega_{\phi}(t))} \leq C(\|\phi\|_{C^{3,\alpha}(0,a)})\cdot\|V_{out}\|_{C^{2,\alpha}(\Gamma_0)} . \tag{6.4}$$

Combining (6.3) and (6.4), we have

$$(6.5) \qquad \left\| \frac{1}{\varepsilon} \int_{\phi}^{b} q^{\varepsilon} d\eta + \sqrt{1 + \phi_x^2} \cdot \frac{\partial p}{\partial n} \right\|_{C(\Gamma)} \leqq O(\sqrt{\varepsilon}) \cdot C(\|\phi\|_{C^{3,\alpha}(0,a)}) \leqq O(\sqrt{\varepsilon}) .$$

Here we used (4.7) in the assumption (B).

6.2. $\phi^{\varepsilon}$ satisfies

$$(3.6) \qquad \frac{\partial \phi^{\varepsilon}}{\partial t} = -d\left(1 - \frac{1}{\varepsilon} \int_{\phi^{\varepsilon}}^{b} p^{\varepsilon}(x,\eta) d\eta\right) + V_{in} .$$

Substracting (6.1) from (3.6) and putting $M^{\varepsilon} = \phi^{\varepsilon} - \phi$, we have

$$(6.6) \qquad \frac{\partial M^{\varepsilon}}{\partial t} = - \frac{d}{\varepsilon} (F(\phi^{\varepsilon}) - F(\phi)) + O(\sqrt{\varepsilon})$$

where $F(\phi^{\varepsilon}) = \int_{\phi^{\varepsilon}}^{b} p^{\varepsilon} d\eta$ and $F(\phi) = \int_{\phi}^{b} q^{\varepsilon} d\eta$. Since there holds

$$F(\phi^{\varepsilon}) - F(\phi) = \int_0^1 \frac{dF}{d\phi} (s\phi^{\varepsilon} + (1 - s)\phi) ds \cdot M^{\varepsilon} ,$$

we may estimate the following Fréchet derivative:

$$(6.7) \qquad \left.\frac{dF}{d\phi}\right|_{\phi=\psi^{\varepsilon}} = \int_{\psi^{\varepsilon}}^{b} \frac{\partial p^{\varepsilon}}{\partial \phi} d\eta - p^{\varepsilon}(\psi^{\varepsilon}) ,$$

where $\psi^{\varepsilon} = s\phi^{\varepsilon} + (1 - s)\phi$ $(s \in (0,1))$. From the assumptions (A) and (B), we see

$$(6.8) \qquad \|\psi^{\varepsilon}\|_{C^{3,\alpha}(0,a)} \leqq K \quad (\alpha \in (0,1)) .$$

Let us recall that $\frac{\partial p^{\varepsilon}}{\partial \phi}$ was defined in proposition 3 of section 5.

6.3. If we put $Q^{\varepsilon} = \frac{\partial p^{\varepsilon}}{\partial \phi}$, then $Q^{\varepsilon}$ satisfies some properties included in propositions 2 and 3. Define $\Gamma^{\varepsilon} = \Gamma^{\varepsilon}(t) = \{(x,y) | 0 < x < a,\ y = \phi^{\varepsilon}(x,t)\}$. Let us estimate (6.7) for $m > 0$,

$$(6.9) \qquad \left\| \int_{\psi^{\varepsilon}}^{b} Q^{\varepsilon} d\eta - p^{\varepsilon}(\psi^{\varepsilon}) \right\|_{H^m(\Gamma^{\varepsilon})}$$

$$\leq \Big\| \sqrt{1 + \psi_x^{\varepsilon 2}} \cdot \Big( \int_{(\Gamma^{\varepsilon})^{\perp}} Q^{\varepsilon} d(\Gamma^{\varepsilon})^{\perp} - \sqrt{\varepsilon}\, Q^{\varepsilon} \Big)$$

$$+ \sqrt{\varepsilon} \sqrt{1 + \psi_x^{\varepsilon 2}} \cdot \Big( Q^{\varepsilon} - \frac{1}{\sqrt{\varepsilon}} \langle n, e_y \rangle p^{\varepsilon} \Big) \Big\|_{H^m(\Gamma^{\varepsilon})}$$

$$\left(\text{note } \sqrt{1 + \psi_x^{\varepsilon 2}} \cdot \langle n, e_y \rangle = 1\right)$$

$$\leq \left\| \sqrt{1 + \psi_x^{\varepsilon 2}} \right\|_{W^{m,\infty}(\Gamma^{\varepsilon})} \cdot \Big( \Big\| \int_{(\Gamma^{\varepsilon})^{\perp}} Q^{\varepsilon} d(\Gamma^{\varepsilon})^{\perp} - \sqrt{\varepsilon}\, Q^{\varepsilon} \Big\|_{H^m(\Gamma^{\varepsilon})}$$

$$+ \sqrt{\varepsilon} \cdot \Big\| Q^{\varepsilon} - \frac{1}{\sqrt{\varepsilon}} p^{\varepsilon} \langle n, e_y \rangle \Big\|_{H^m(\Gamma^{\varepsilon})} \Big)$$

$$\leq O(\varepsilon) \left\| \sqrt{1 + \psi_x^{\varepsilon 2}} \right\|_{W^{m,\infty}(\Gamma^{\varepsilon})} \cdot \| \langle n, e_y \rangle \|_{W^{m+1,\infty}(\Gamma^{\varepsilon})} \cdot \left\| \frac{\partial p}{\partial n} \right\|_{H^{m+1}(\Gamma^{\varepsilon})}$$

(by (5.18) and (5.19)). Putting $m = 1$ in the above inequality,

$$(6.10) \qquad \left\| \int_{\psi^{\varepsilon}}^{b} Q^{\varepsilon} d\eta - p^{\varepsilon}(\psi^{\varepsilon}) \right\|_{C(\Gamma^{\varepsilon})} \leq \varepsilon \cdot C\left(\|\psi^{\varepsilon}\|_{W^{3,\infty}(0,a)}\right) \cdot \left\| \frac{\partial p}{\partial n} \right\|_{H^2(\Gamma^{\varepsilon})}$$

$$\leq \varepsilon \cdot C\left( \|\psi^{\varepsilon}\|_{C^{3,\alpha}(0,a)} \right) \leq \varepsilon K .$$

Here we used the same discussions as in 6.1 and note that $K$ is independent of $\varepsilon \in (0,1)$.

6.4. Integrating (6.6) in $(0,t)$, we have

$$M^{\varepsilon} = -\frac{d}{\varepsilon}\int_0^t (F(\phi^{\varepsilon}) - F(\phi))d\tau + O(\sqrt{\varepsilon})\cdot t \tag{6.11}$$

from which

$$\|M^{\varepsilon}(\cdot,t)\|_{C(0,a)} \leq \frac{d}{\varepsilon}\int_0^t \left\|\frac{dF}{d\phi}\Big|_{\phi=\psi^{\varepsilon}}\right\|_{C(0,a)}\cdot\|M^{\varepsilon}(\cdot,\tau)\|_{C(0,a)}d\tau + O(\sqrt{\varepsilon})\cdot t \ . \tag{6.12}$$

Substituting (6.10) into (6.12), we have

$$\|M^{\varepsilon}(\cdot,t)\|_{C(0,a)} \leq d\cdot K\int_0^t \|M^{\varepsilon}(\cdot,\tau)\|_{C(0,a)}d\tau + O(\sqrt{\varepsilon}) \ . \tag{6.13}$$

from which

$$\|M^{\varepsilon}(\cdot,t)\|_{C(0,a)} \leq O(\sqrt{\varepsilon}) \quad \text{for } \forall\, t \in [0,T] \ . \tag{6.14}$$

Therefore

$$\max_{t\in[0,T]} \|M^{\varepsilon}(\cdot,t)\|_{C(0,a)} \to 0 \quad \text{as } \varepsilon \to 0 \ . \tag{6.15}$$

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Drainage of the hearth is an important component of a successful operation of a blast furnace. In order to investigate the influence of tapping conditions due to the shape of the slag surface, the one-phase flow of slag during tapping was modeled as a free boundary problem. This problem was reformulated by using the method of integrated penalty and, then was simulated by using a finite difference method developed by the author

(cont.)

ABSTRACT (cont.)

The objective of this report is to give a mathematical justification of the penalty method formulation, in which the perturbation with respect to the domain and the asymptotic properties of solutions of the boundary value problem for an elliptic equation with penalty terms are used.